functions are the only continuous solutions, was first proved by A. L. Cauchy [1

Proof. (i) Let f LIF($\mathbb{R}^n$) and g AD($\mathbb{R}^n$). Fix $x_1, \ldots, x_k$ $\mathbb{R}^n$ and $q_1, \ldots q_k$
$\mathbb{Q}$

This choice is possible since

$$\left|\bigcup_{f\in F}((h+f)[\{x_\xi : \xi<\alpha\}] \cup \{f(x_\alpha)\})\right| \leq (\alpha+1)|F| < \mathfrak{c}.$$

It is easy to see that all the required properties of h are preserved. This ends the proof of A(LIF(R

Notice here that $A(\mathrm{LIF}) = \mathfrak{c}$ (Fact 2.3 (v)) implies, in particular, that every function from $\mathbb{R}^{\mathbb{R}}$ can be written as the algebraic sum of two linearly independent functions. In other words $\mathrm{LIF} + \mathrm{LIF} = \mathbb{R}^{\mathbb{R}}$. Since we only found the upper bound for $A(\mathrm{HF})$, it wo16Td[(can)-347(b)-27spper bo12313(A)-350((r)v(e)-350d[(y50((r))1(d))-1(.er)-470(tn)1(dg50((r

Proof. Notice first that if $|LC(f, 2)| = \mathfrak{c}$ then case (a) holds with $Z = \{0$

From (•) we see that if $\mathrm{Lin}_{\mathrm{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathrm{Q}}(X) = \{0\}$ holds for c-many ξ then the set Z satisfies the condition $|\ \ _{z\in Z} \mathrm{LC}(f, 2, z)| = \mathfrak{c}$. Obviously $Z \quad [\mathrm{R}^n]^{<\mathfrak{c}}$. Thus, case (a) holds.

Summarizing the above discussion, we just need to consider a situation when $\dim(\{x_1^\xi, x_2^\xi, x_3^\xi\}) = 2$ and $\mathrm{Lin}_{\mathrm{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathrm{Q}}(X) = \{0\}$ for all ξ. Recall that $q_1x_1^\xi + q_2x_2^\xi + q_3x_3^\xi = 0$, where $q_1, q_2, q_3 \quad \mathrm{Q} \setminus \{0\}$. If two of $x_1^\xi, x_2^\xi, x_3^\xi$ were dependent over Q then we would have $\dim(\{x_1^\xi, x_2^\xi, x_3^\xi\}) \leq 1$. Thus, $x_1^\xi, x_2^\xi, x_3^\xi$ are pairwise independent. Now it is easy to see that case (b) holds. ∎

Lemma 3.8. *Let* $X \quad [\mathrm{R}^n]^{<\mathfrak{c}}$, x / X, *and* $y \quad \mathrm{R}$. *Suppose also that* $h, g\colon X \to \mathrm{R}$ *are functions linearly independent over* Q. *Then there exist extensions* h', g' *of* h *and* g *onto* $X \quad \{x\}$ *such that* h' *and* g' *are linearly independent over* Q *and* $h'(x) + g'(x) = y$.

Proof. Choose $h'(x) \quad \mathrm{R} \setminus \mathrm{Lin}_{\mathrm{Q}}(h[X] \quad g[X] \quad \{y\})$. This choice is possible since $|\mathrm{Lin}_{\mathrm{Q}}(h[X] \quad g[X] \quad \{y\})| < \mathfrak{c}$. Then define $g'(x) = y - h'(x)$. It is easy to d[(x)]TF89.963TfmF(x)]T3Tf-301.463

holds because $f(-a_0) + f(a_0) = c$ and $m_0 = c$ if $c = 0$. Thus $0, c$, $0, m_0$ $\mathrm{Lin}_{\mathbb{Q}}(h|A_0) \cap \mathrm{Lin}_{\mathbb{Q}}(g|A_0)$. It is easily seen that $h|A_0$ and $g|A_0$ satisfy (a) and (b).

$x_\xi \in \mathrm{dom}(h_\xi) = \mathrm{dom}(g_\xi)$ and $v_\xi \in \mathrm{Lin}_{\mathbb{Q}}(h_\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(g_\xi)$, where h_ξ and g_ξ denote the extensions obtained in the step ξ.

Let $\beta < \mathfrak{c}$. Assume that $v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(\bigcup_{\xi<\beta} h_\xi) \cup \mathrm{Lin}_{\mathbb{Q}}(\bigcup_{\xi<\beta} g_\xi)$. Choose an $a \in \mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}(\mathrm{dom}(\bigcup_{\xi<\beta} h_\xi))$ and define $h(x)$ by $0, h(x) = \frac{1}{2} v_\beta$ for $x \in \{-a, a\}$. Put also $g(x) = f(x) - h(x)$. Since $f(-a) + f(a) \in \mathrm{LC}(f)$, (3.3) implies that $v_\beta \in \mathrm{Lin}_{\mathbb{Q}}(x$

g

h gnd

The inductive construction of functions h and g is somewhat similar to the one from the previous case. So assume that $\beta < \mathfrak{c}$ and the construction has been carried out for all $\xi < \beta$. If $v_\beta \ / \ \mathrm{Lin}_{\mathbb{Q}}(h)$ then let $X = \mathrm{dom}(h) = \mathrm{dom}(g)$ and $Y \quad [\mathbb{R}]^{<\mathfrak{c}}$ be such a set that $\mathrm{Lin}_{\mathbb{Q}}(g \quad \{v_\beta\}) \quad \mathbb{R}^n \times Y$. By Property 2 (b), there exist $p_1, p_2, p_3 \quad \mathbb{Q} \setminus \{0\}$ and pairwise independent $x_1, x_2, x_3 \quad \mathbb{R}^n$ such that $\sum_1^3 p_i x_i = 0$, $\mathrm{Lin}_{\mathbb{Q}}(x_1, x_2, x_3) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$, and $\sum_1^3 p_i f(x_i) \ / \ Y$.

We extend h and g onto $\{x_1, x_2, x_3\}$. Choose $h(x_1), h(x_2), h(x_3) \quad \mathbb{R}$ in such a way that

$$\sum_1^3 p_i \left(x_i, h(x_i) \right) = 0, \quad \sum_1^3 p_i h(x_i) = v_\beta .$$

Then put $g(x_i) = f(x_i) - h(x_i)$ for $i \leq 3$. Obviously v_βgLin$_{\mathbb{Q}}$(h) pnd hv+]TF119.963Tf10.